THE PLANE DIRICHLET PROBLEM FOR CERTAIN MULTIPLY CONNECTED REGIONS

By

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1. Introduction

It is the purpose of this paper to show, by introducing the concept of reflection, how we can simplify the problem of finding a solution to the Dirichlet problem for certain multiply-connected regions.

We consider the problem of finding a single-valued function harmonic in a connected region S^+ bounded by nonintersecting contours $L_0, L_1, \cdots, L_p$ for which L_0 contains all the others. We use the notation of Muskhelishvili [2] and Mikhlin [1]. Let $L = L_0 \cup L_1 \cup \cdots \cup L_p$, S_j^- be the finite region bounded by L_j, $j = 1, 2, \cdots, p$ and S_0^- be the infinite region bounded by L_0. The positive direction of L is taken so that S^+ lies to the left of L. We assume L_0 is analytic and satisfies certain additional assumptions and that the L_j, $j = 1, \cdots, p$ have continuous curvature, see Fig. 1.

We shall show that the Dirichlet problem for S^+ can be reduced to an integral equation not involving L_0, or equivalently, can be reduced to p integral equations for p unknowns, one on each L_j, $j = 1, \cdots, p$. We shall show that this integral equation (system) has a unique solution in general. This contrasts to the usual process, in that there is a system of $p + 1$ integral equations with $p + 1$ unknowns to determine. In the case of a doubly-connected region, this reduces the problem to the solution of only one integral equation in one unknown, instead of a coupled system of two equations in two unknowns.

As reflection plays such a key role in the development, we shall give some examples of analytic curves of the type specified and shall show the

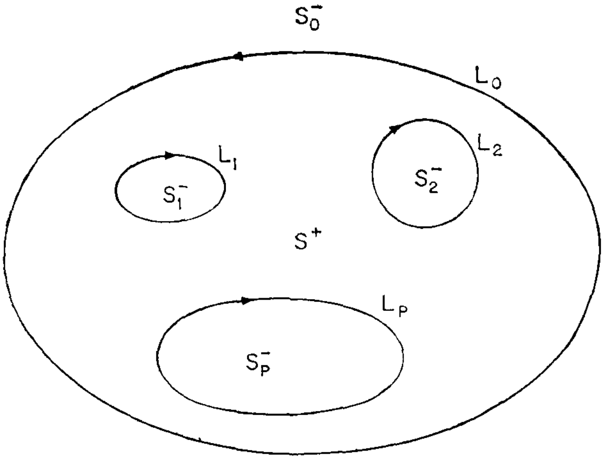

FIG. 1.

reflection of curves in these curves. These results were worked out on an on-line computer with the help of Dan Zuras to whom *I* wish to express my gratitude.

2. Formulation of the problem

We shall restrict ourselves to curves L_0 of the form:

$$\left.\begin{aligned} x(\theta) &= \sum_{k=0}^{n} a_k \cos k\theta + b_k \sin k\theta \\ y(\theta) &= \sum_{k=0}^{m} \alpha_k \cos k\theta + \beta_k \sin k\theta \end{aligned}\right\} \quad 0 \leqq \theta < 2\pi, \tag{2.0}$$

with $x'^2(\theta) + y'^2(\theta) \neq 0$, $(a_n, b_n) \neq (0,0) \neq (\alpha_m, \beta_m)$ and if $n = m$ then either $\alpha_n^2 + \beta_n^2 \neq a_n^2 + b_n^2$ or $\alpha_n a_n + \beta_n b_n \neq 0$. We also require that if

$$f(t) = \bar{c}_n + \bar{c}_{n-1}\bar{t} + \cdots + \bar{c}_1\bar{t}^{n-1} + [2a_0 - z - \zeta]\bar{t}^n + c_1\bar{t}^{n+1} + \cdots + c_n\bar{t}^{2n}$$

$$g(t) = \bar{\gamma}_m + \bar{\gamma}_{m-1}\bar{t} + \cdots + \bar{\gamma}_1\bar{t}^{m-1}$$

$$+ [2\alpha_0 - i(z - \zeta)]\bar{t}^m + \gamma_1\bar{t}^{m+1} + \cdots + \gamma_m\bar{t}^{2m}$$

where

$$c_k = a_k + ib_k, \qquad \gamma_k = \alpha_k + i\beta_k$$

and if

$$M(z,\zeta) = \text{resultant of } f \text{ and } g = R[f,g]$$

and if

$$P[z] = R[M(z,\zeta),\ M_\zeta(z,\zeta)]$$

$$Q[z] = R[M(z,\zeta),\ M_z(z,\zeta)]$$

then no zero of P or Q lies in $L \cup S^+$.

We are looking for a real single-valued function $u(z)$, $z = x + iy$, harmonic in S^+, continuous in $S^+ \cup L$ and such that

$$(2.1) \qquad u(z) \to F(t) \quad \text{as} \quad z \to t \quad \text{on} \quad L, \quad z \in S^+,$$

i.e.,

$$(2.1.1) \qquad u(z) \to F_j(t_j) \quad \text{as} \quad z \to t_j \quad \text{on} \quad L_j, \quad j = 0, 1, \cdots, p,$$

in which F and F_j are Hölder continuous.

Before we can formulate the integral equation, we need to recall certain facts about geometrical reflection. It is shown in [3] that if L_0 is of the form we have, then there exists a *reflection function* $G(z)$ for L_0, single-valued and analytic in Ω such that if $z \in S^+$ and

$$\hat{z} = \overline{G(z)}$$

then for z close enough to L_0, $\hat{z} \notin S^+$. We shall assume that for all $z \in S^+$, $\hat{z} \notin S^+$. Then

(i) $\hat{z} \in S_0^-$;

(ii) $\hat{z} = z$ for z on L_0;

(iii) $G(z)$ can be extended as a single-valued function analytic on $S^+ \cup L \cup \hat{S}^+$ where

$$\hat{S}^+ = \{\zeta : \zeta = \hat{z}, \quad z \in S^+\} \equiv \overline{G(S^+)};$$

(iv) $G'(z) \neq 0$ for $z \in S^+ \cup L_0 \cup \hat{S}^+$;

(v) $\hat{\hat{z}} = z$ for $z \in S^+ \cup L \cup \hat{S}^+$.

We look for

$$u(z) = \operatorname{Re} \phi(z). \tag{2.2}$$

Even though $u(z)$ is single-valued, $\phi(z)$ is not single-valued. In general, however,

$$\phi(z) = \phi^*(z) + \sum_{k=1}^{p} A_k \ln(z - z_k) \tag{2.3}$$

in which the A_k are real constants to be determined, z_k is any point in S_k^-, $k = 1, \cdots, p$ and $\phi^*(z)$ is single-valued.

We seek $\phi^*(z)$ of the form

$$\phi^*(z) = \frac{1}{2\pi i} \int_L \mu(t) \left\{ \frac{1}{t - z} - \frac{1}{t - \hat{z}} \right\} dt \tag{2.4}$$

in which $\mu(t)$ is a real single-valued Hölder continuous function on L.

We consider each of the $p + 1$ integrals in (2.4), i.e.,

$$\phi_k^*(z) = \frac{1}{2\pi i} \int_{L_k} \mu_k(t) \left\{ \frac{1}{t - z} - \frac{1}{t - \hat{z}} \right\} d\zeta, \quad k = 0, 1, \cdots, p. \tag{2.5}$$

By (ii), we see

$$\phi_k^*(z) \rightarrow 0 \quad \text{as} \quad z \rightarrow t_0 \quad \text{on} \quad L_0 \quad \text{for} \quad k = 1, 2, \cdots, p. \tag{2.6}$$

By the Plemelj formulas [2], we see that

$$\phi_0^*(z) \rightarrow \mu_0(t_0) \quad \text{as} \quad z \rightarrow t_0 \quad \text{on} \quad L_0, \tag{2.7}$$

and that

$$\phi_k^*(z) \rightarrow +\tfrac{1}{2}\mu_k(t_k) + \frac{1}{2\pi i} \int_{L_k} \mu_k(t) \left\{ \frac{1}{t - t_k} - \frac{1}{t - \hat{t}_k} \right\} dt \tag{2.8}$$

as $z \to t_k \in L_k$. Thus we arrive at the integral equations:

$$\mu_0(t_0) = F_0(t_0) - \sum_{k=1}^{p} A_k \ln|t_0 - z_k| \tag{2.9}$$

$$\mu(t^*) + \operatorname{Re}\left[\frac{1}{\pi i}\int_L \mu(t)\left\{\frac{1}{t-t^*} - \frac{1}{t-\hat{t}^*}\right\}dt\right] = 2F(t^*) - 2\sum_{k=1}^{p} A_k \ln|t^* - z_k| \tag{2.10}$$

and combining yields the integral equation:

$$+\mu(t^*) + \operatorname{Re}\left[\frac{1}{\pi i}\int_{\mathscr{L}} \mu(t)\left\{\frac{1}{t-t^*} - \frac{1}{t-\hat{t}^*}\right\}dt\right] = H(t^*), \quad t^* \in \mathscr{L} \tag{2.11}$$

where

$$\mathscr{L} = L_1 \cup L_2 \cup \cdots \cup L_p$$

and

$$H(t^*) = 2F(t^*) - 2\sum_{k=1}^{p} A_k \ln|t^* - z_k| - \operatorname{Re}\frac{1}{\pi i}\int_{L_0}\left[F_0(t) - \sum_{k=1}^{p} A_k \ln|t - z_k|\right]$$

$$\times\left\{\frac{1}{t-t^*} - \frac{1}{t-\hat{t}^*}\right\}dt. \tag{2.12}$$

Reflection has permitted us to solve one of the $p+1$ integral equations explicitly, leaving only p equations to be solved.

3. Solution of the problem

We wish to show that (2.11) has a Hölder continuous solution and it is unique. Toward this end we wish to use the Fredholm alternative. Thus we shall consider instead of (2.11), the integral equation

$$v(t^*) = \lambda \operatorname{Re}\frac{1}{\pi i}\int_{\mathscr{L}} v(t)\left\{\frac{1}{t-t^*} - \frac{1}{t-\hat{t}^*} + k(t,t^*)\frac{i}{t'}\right\}dt \tag{3.1}$$

where

$$k(t,t^*)\left\{\begin{array}{l} = 1 \text{ if } t,t^* \text{ lie on same } L_k \\ \\ = 0 \text{ if } t,t^* \text{ lie on different } L_k \end{array}\right\} \text{ where } t' = \frac{dt}{ds}, s = \text{arc length}$$

and prove:

Theorem 1. *$\lambda = -1$ is not an eigenvalue of* (3.1).

Proof. If $\lambda = -1$ were an eigenvalue and $\sigma(t)$ a corresponding Hölder continuous eigenfunction (real), then

$$\sigma(t^*) = -\operatorname{Re}\frac{1}{\pi i}\int_{\mathcal{L}} \sigma(t)\left\{\frac{1}{t-t^*} - \frac{1}{t-\hat{t}^*} + k(t,t^*)\frac{i}{t'}\right\} dt. \tag{3.2}$$

Let

$$-b_k = \frac{1}{\pi}\int_{L_k} \sigma(t)ds, \qquad k = 1,2,\cdots,p. \tag{3.3}$$

Then (3.2) can be written:

$$-\sigma(t^*) = \operatorname{Re}\left\{\frac{1}{\pi i}\int_{\mathcal{L}} \sigma(t)\left\{\frac{1}{t-t^*} - \frac{1}{t-\hat{t}^*}\right\}dt\right\} - b_k, \quad k = 1,2,\cdots,p \tag{3.4}$$

for t^* on L_k.

Consider the function

$$\alpha(z) = \frac{1}{\pi i}\int_{\mathcal{L}} \sigma(t)\left\{\frac{1}{t-z} - \frac{1}{t-\hat{z}}\right\} dt \tag{3.5}$$

which is harmonic in $S^+ \cup L_0 \cup S_0^-$. Using the Plemelj formulas and letting $z \to t^*$ on $\mathcal{L}$ gives upon utilizing (3.4):

$$\operatorname{Re}\alpha(t^*) = +\sigma(t^*) + \operatorname{Re}\frac{1}{\pi i}\int_{\mathscr{L}} \sigma(t)\left\{\frac{1}{t-t^*} - \frac{1}{t-\hat{t}^*}\right\} dt = b_k,$$

for t^* on L_k, $k = 1, 2, \cdots, p$.
But on L_0,

$$\alpha(t_0) = 0.$$

Let

$$\psi(z) = \frac{1}{\pi i}\int_{\mathscr{L}} \sigma(t)\frac{1}{t-z}\, dt.$$

Then

$$\alpha(z) = \psi(z) - \psi(\hat{z})$$

and

$$\operatorname{Re}\alpha(z) = \operatorname{Re}\chi(z)$$

where

$$\chi(z) = \psi(z) - \overline{\psi(\hat{z})}$$

is analytic for z in $S^+ \cup L_0 \cup \hat{S}_+$. But

$$\begin{aligned} \operatorname{Re}\alpha(z) = \operatorname{Re}\chi(z) &= b_k \quad \text{on} \quad L_k \\ &= 0 \quad \text{on} \quad L_0. \end{aligned}$$

Thus by the Cauchy-Riemann equations:

$$\frac{\partial\{\operatorname{Im}\chi(z)\}}{\partial n} = -\frac{\partial\{\operatorname{Re}\chi(z)\}}{\partial s} = 0 \text{ on } \mathscr{L} \cup L_0,$$

where n = outer normal (provided the orientation of (n, t) is the same as (x, y)). But $\operatorname{Im}\chi(z)$ is harmonic in $S^+ \cup L_0 \cup S_0^-$ and thus by the uniqueness of the solution of the Neumann problem

$$\operatorname{Im}\chi(z) = \text{constant in } S^+$$

and thus

$$\chi(z) = \text{constant in } S^+.$$

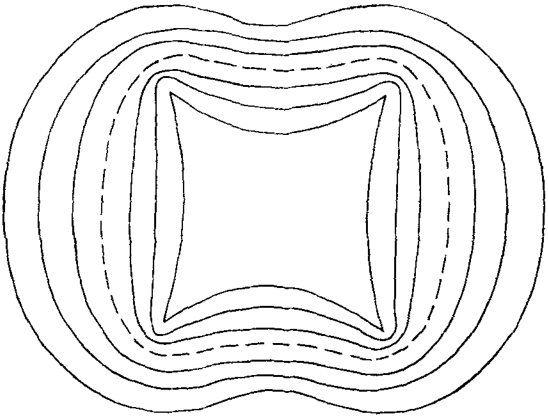

FIG. 2.

$$x(\sigma) = \cos\sigma$$
$$y(\sigma) = \sin\sigma + .2\sin 3\sigma$$
$$\sigma = \alpha + i\beta, \quad 0 \leqq \alpha \leqq 2\pi, \quad \beta = 0, \pm .1, \pm .2, \pm .3$$

But

$$\operatorname{Re}\chi(z) = 0 \text{ on } L_0,$$

thus

$$\chi(z) = ia \text{ in } S^+, \quad a = \text{real constant.}$$

Hence

$$\operatorname{Re}\alpha(z) = 0 \text{ in } S^+$$

and

$$b_k = 0.$$

Also we have

$$ia = \psi(z) - \overline{\psi(\hat{z})} \text{ in } S^+ \cup L \cup \hat{S}_+$$

and thus

$$\psi(z) - \overline{\psi(z)} = ia \text{ on } L_0,$$

i.e.,

$$\operatorname{Im}\psi(z) = \tfrac{1}{2}a \text{ on } L_0.$$

But $\psi(z)$ is analytic in $S^+ \cup L_0 \cup S_0^-$ and thus $\operatorname{Im}\psi(z)$ is harmonic in $S^+ \cup L_0 \cup S_0^-$. Moreover $\operatorname{Im}\psi(z)$ takes on the value $\frac{1}{2}a$ on L_0 and tends uniformly to zero as $z \to \infty$. Thus

$$\psi(z) = \tfrac{1}{2}(\bar{a} + ia) \text{ constant in } S^+ \cup L_0 \cup S_0^-.$$

Thus

$$\frac{1}{\pi i}\int_{\mathscr{L}} [\sigma(t) - \tfrac{1}{2}\bar{a}]\frac{dt}{t-z} = 0 \text{ for all } z \text{ in } S^+ \cup L_0 \cup S_0^-,$$

and by Harnack's theorem we can conclude that

$$\sigma(t) = a_0 + \tilde{c}_k, \qquad k = 1, 2, \cdots, p,$$

$\tilde{c}_k$ = constant for t on L_k.

But

$$0 = -b_k = \frac{1}{\pi}\int_{L_k} \sigma(t)\,ds = \frac{1}{\pi}[a_0 + \tilde{c}_k]\,l_k, \qquad l_k = \text{length of } L_k$$

and thus

$$a_0 + \tilde{c}_k = 0$$

and finally

$$\sigma(t) \equiv 0$$

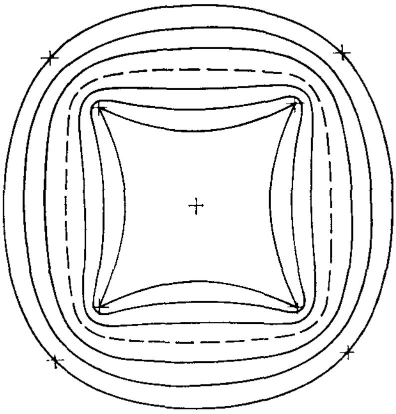

FIG. 3.
$x(\sigma) = \cos\sigma + .1\cos 3\sigma$
$y(\sigma) = \sin\sigma + .1\sin 3\sigma$
$\sigma = \alpha + i\beta, \quad 0 \leqq \alpha \leqq 2\pi, \quad \beta = 0, \pm .1, \pm .2, \pm .3$
The crosses represent the singular points

which shows that $\lambda = -1$ is not an eigenvalue and the theorem is proved.

We now return to the original equation (2.11) and prove

Theorem 2. *There exist* A_k *such that* (2.11) *has a unique solution for all Hölder continuous* $H(t)$.

Proof. We consider equation

$$v(t^*) = -\operatorname{Re}\frac{1}{\pi i}\int_{\mathscr{L}} v(t)\left\{\frac{1}{t-t^*} - \frac{1}{t-\hat{t}^*} + k(t,t^*)\frac{i}{t'}\right\} dt + H(t^*). \tag{3.6}$$

This equation in general has a unique solution by Theorem 1. The solution will depend on the parameters $A_1, A_2, \cdots, A_p$. If we impose on the solution the conditions

$$-b_k = \frac{1}{\pi}\int_{L_k} v(t)ds = 0, \qquad k = 1,2,\cdots,p, \tag{3.7}$$

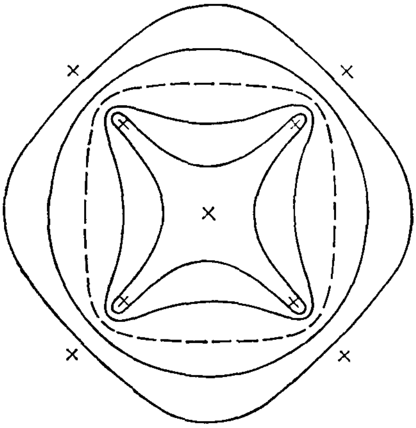

FIG. 4.

$x(\sigma) = \cos\sigma + .1\cos 3\sigma$
$y(\sigma) = \sin\sigma + .1\sin 3\sigma$
$\sigma = \alpha + i\beta, \quad 0 \leqq \alpha \leqq 2\pi, \quad \beta = b\cos^2 2\alpha, \quad 0 \leqq \alpha \leqq 2\pi, \quad b = \pm .1, \pm .2$

then we obtain the solution of equation (2.11). Thus the problem reduces to showing that the solution of (3.6) subject to the constraints (3.7) has a unique solution.

The solution of (3.6) can be written in terms of the resolvent

$$v(t^*) = \int_{\mathscr{L}} \Gamma(t^*, t) H(t)\, dt$$

(3.8)

$$= \int_{\mathscr{L}} \Gamma(t^*, t)\left[\mathscr{F}(t) + \sum_{k=1}^{p} A_k \mathscr{G}(t, z_k)\right] dt$$

where

$$\mathscr{F}(t) = +2F(t) - \operatorname{Re}\frac{1}{\pi i}\int_{L_0} F_0(t_0)\left\{\frac{1}{t_0 - t} - \frac{1}{t_0 - \hat{t}}\right\} dt_0$$

$$\mathscr{G}(t) = -2\ln|t - z_k| + \operatorname{Re}\frac{1}{\pi i}\int_{L_0} \ln|t_0 - z_k|\left\{\frac{1}{t_0 - t} - \frac{1}{t_0 - \hat{t}}\right\} dt_0. \tag{3.9}$$

Imposing the conditions (3.7) on (3.8) leads to a system of equations

$$b_j = 0 = f_j + \sum_{k=1}^{p} g_{jk} A_k, \qquad j = 1, 2, \cdots, p, \tag{3.10}$$

where it is clear what f_j and g_{kj} are. We wish to show (g_{jk}) is nonsingular. If it were singular then for $F(t) = 0$, i.e., $f_j = 0$, we would have a nontrivial solution $(A_1^0, A_2^0, \cdots, A_p^0)$ of (3.10). But then

$$u(z) = \operatorname{Re}\{\phi^*(z)\} + \sum_{k=1}^{p} A_k^0 \ln|z - z_k|$$

would give a nontrivial solution to the Dirichlet problem in S^+ with zero boundary data which is impossible. To see that we cannot have

$$\operatorname{Re}\{\phi^*(z)\} + \sum_{k=1}^{p} A_k^0 \ln |z - z_k| \equiv 0 \quad \text{in} \quad S^+.$$

We note if we did then

$$\phi^*(z) + \sum_{k=1}^{p} A_k^0(z - z_k) = \text{constant in } S^+$$

which is impossible since $\phi^*(z)$ is a single-valued function in S^+. Thus g_{jk} is nonsingular and the theorem is established.

4. Examples of reflection

In this section we look at several examples of possible geometries that were worked out using an on-line computer. I wish to acknowledge the great help given in the computations by Dan Zuras. The pictures were photographed with a polaroid camera directly from the scope, the photographs were photographed to obtain a negative and the negatives were enlarged to the proper size. The enlargement was then traced to give the pictures of the text.

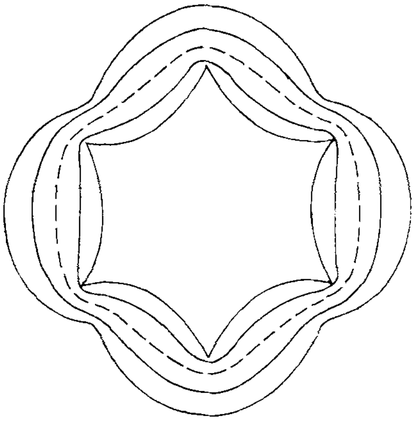

FIG. 5.

$x(\sigma) = \cos\ \sigma$
$y(\sigma) = \sin\ \sigma + .1\ \sin\ 5\sigma$

$0 = \alpha + i\beta, \qquad 0 \leqq \alpha \leqq 2\pi, \qquad \beta = 0, \pm .1, \pm .2$

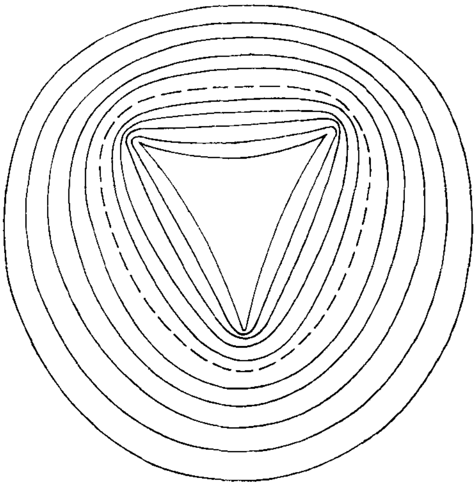

FIG. 6.

$$x(\sigma) = \cos\sigma + .2\cos 2\sigma$$
$$y(\sigma) = \sin\sigma$$
$$\sigma = \alpha + i\beta, \qquad 0 \leqq \alpha \leqq 2\pi, \qquad \beta = 0, \pm .1, \pm .2, \pm .3, \pm .4, \pm .5$$

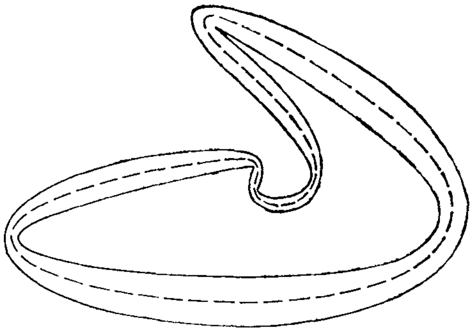

FIG. 7.

$$x(\sigma) = (.60 + .85i)\cos\sigma - (+.74 + .43i)\cos 2\sigma + (.82 - .074i)\cos 3\sigma$$
$$y(\sigma) = (-.40 + .63i)\sin\sigma - (.77 + .98i)\sin 2\sigma + (.034 + .053i)\sin 3\sigma$$
$$\sigma = \alpha + i\beta, \qquad 0 \leqq \alpha \leqq 2\pi, \qquad \beta = 0, \pm .05$$

In each example the dashed curve is the analytic curve whose equations are given below. The closest interior curve to the dashed curve is reflected to the closest exterior curve to the dashed curve, the second closest interior curve is reflected to the second closest exterior curve, etc.

The reflection function was defined in [3] by setting

$$x = \tfrac{1}{2}(z + \zeta), \qquad y = \frac{1}{2i}(z - \zeta)$$

in (2.0), solving for

$$\zeta = G(z)$$

and setting

$$\hat{z} = \overline{G(z)}.$$

In the computations for a reflection curve through the curve L_0:

$$\left.\begin{aligned} x(\alpha) &= \sum_{k=0}^{n} a_k \cos k\alpha + b_k \sin k\alpha \\ y(\alpha) &= \sum_{k=0}^{m} \alpha_k \cos k\alpha + \beta_k \sin k\alpha \end{aligned}\right\} 0 \leqq \alpha < 2\pi$$

we took

$$x(\sigma) + iy(\sigma) = x[\alpha + i\beta(\alpha)] + iy[\alpha + i\beta(\alpha)] \qquad 0 \leqq \alpha < 2\pi$$

for the curve and

$$x(\bar{\sigma}) + iy(\bar{\sigma}) = x[\alpha - i\beta(\alpha)] + iy[\alpha - i\beta(\alpha)] \qquad 0 \leqq \alpha < 2\pi$$

for the reflected curve.

In order to justify this, let

$$z = H(\sigma) = x(\sigma) + iy(\sigma)$$

and

$$\tilde{z} = H(\bar{\sigma})$$

where we restrict ourselves to regions where $H(\sigma)$ is a 1–1 map, i.e., $x'(\sigma) + iy'(\sigma) \neq 0$. Thus we can solve for

$$\sigma = H^{-1}(z)$$

and obtain

$$\tilde{z} = H[\overline{H^{-1}(z)}].$$

But for z on L_0 we have

$$\tilde{z} = z$$

and thus

$$z = H[\overline{H^{-1}(z)}] \quad \text{for } z \text{ on } L_0.$$

Consider

$$\hat{z} - \tilde{z} = \overline{G(z)} - H[\overline{H^{-1}(z)}].$$

Then $\bar{\hat{z}} - \bar{\tilde{z}} = 0$ on L_0 and is analytic in a neighborhood of L_0 and thus

$$\hat{z} = \tilde{z}$$

and the procedure is justified.

It should also be mentioned that the singular points (+) of G for Figure 4 were found by finding the zeros of the polynomial

$$x'(\bar{\sigma}) + iy'(\bar{\sigma}) = 0 \text{ and } x'(\sigma) - iy'(\sigma) = 0.$$

This is valid since

$$\hat{z} = x(\bar{\sigma}) + iy(\bar{\sigma}) = \overline{G[x(\sigma) + iy(\sigma)]} = \overline{G(z)}$$

and thus

$$x'(\bar{\sigma}) + iy'(\bar{\sigma}) = \overline{G'[z]}\ \overline{x'(\sigma) + iy'(\sigma)}.$$

In all of the figures, $\beta = 0$ gives the curve of reflection.

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